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Introduction to Multi-Objective Optimization

Chapter 40 - LIONBook

Brief recap

Goal: find the **optimal point(s)** of a model, for which **no other point is better**

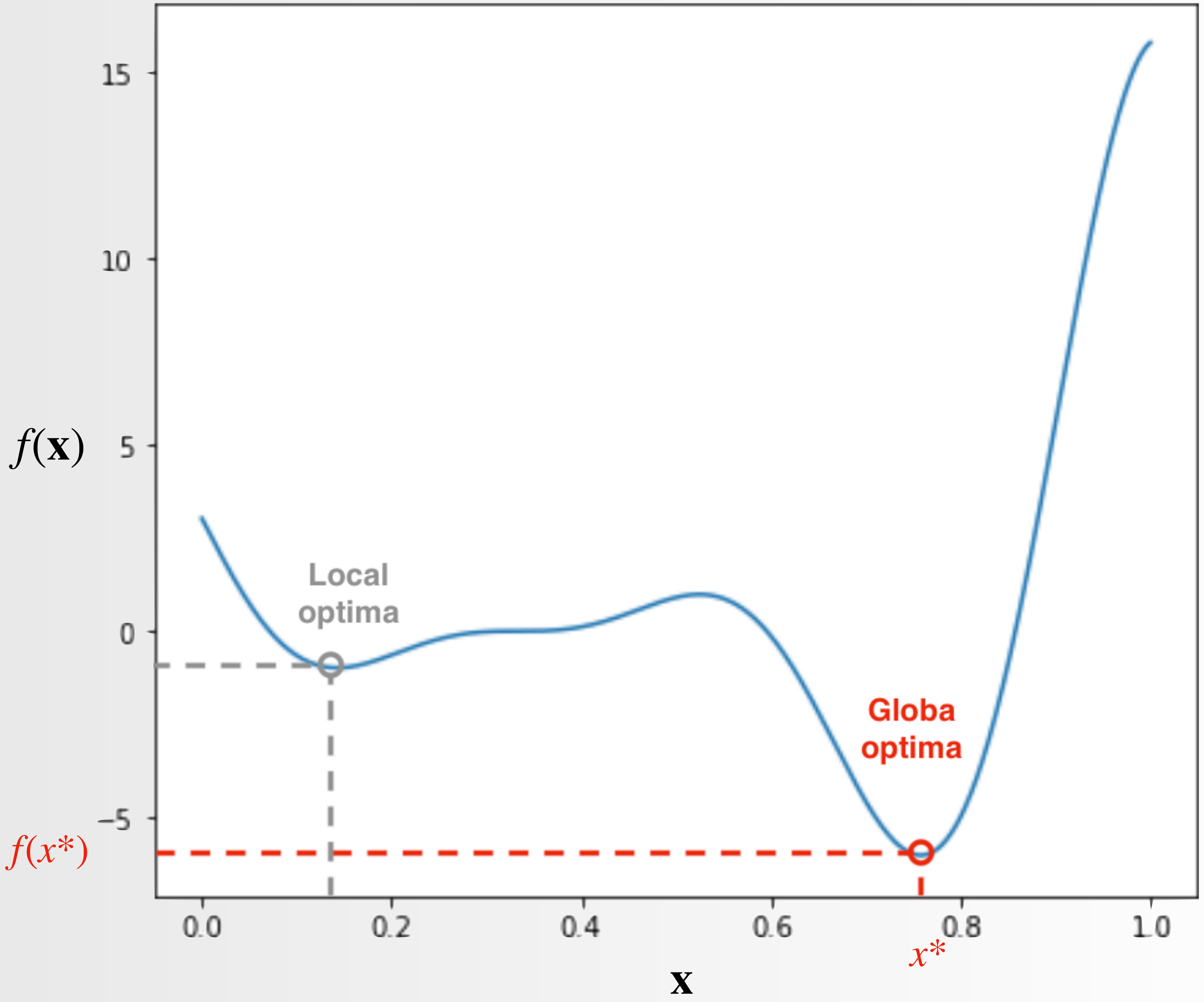
Ω is the set of possible points

$$x^* : f(x^*) \leq f(x), \quad \forall x \in \Omega$$

(or $\geq$)

More generally:

$$\text{minimize/maximize } f(\mathbf{x})$$
$$\text{subject to } \mathbf{x} \in \Omega,$$



Brief recap

Example

$x_1, x_2 \in \Omega$ and $f(\mathbf{x}) : R^n \to R$

Suppose:

- minimization task
- $f(x_1) = 10,\ f(x_2) = 30$

Can we determine which is the **better solution** between the two?

$f(x_1) < f(x_2)$ therefore x_1 $\to$ **trivial**

Multi-Objective optimization

Example

$x_1, x_2 \in \Omega$ and $f(\mathbf{x}) = \{f_1(\mathbf{x}), f_2(\mathbf{x})\} : R^n \rightarrow R^2$

(objectives)

Suppose:

- minimize $f_1(\mathbf{x})$ and maximize $f_2(\mathbf{x})$ at the same time
- $f(x_1) =$ [15, 15] and $f(x_2) =$ [30, 30]

Can we determine which is the **better solution** between the two?

> <u>Not</u> trivial

Multi-Objective optimization

Mathematical formulation

Statement:

$$\text{minimize } \mathbf{f}(\mathbf{x}) = \{f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})\}$$
$$\text{subject to } \mathbf{x} \in \Omega$$

where:

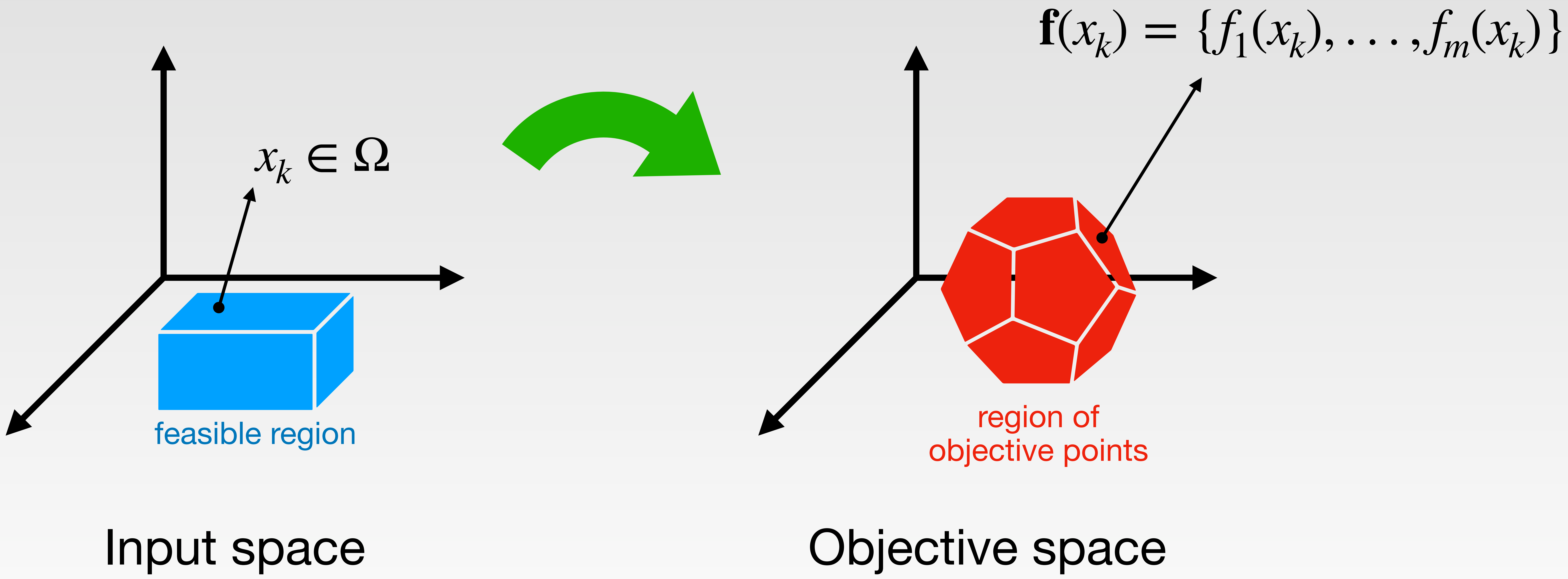
$\mathbf{x} \in R^n$ are the variables and $\mathbf{x} \in \Omega$ feasible region

$\mathbf{f} : \Omega \to R^m$ is made of m **objective functions**

As anticipated before, the problem is **ill-posed** when the objective functions are conflicting, it is not possible to optimize the objectives independently

Multi-Objective optimization

Mathematical formulation



Multi-Objective optimization

For a non-trivial multi-objective optimization problem, objectives are **conflicting** and it is not possible to find a solution that optimize all objectives at the same time.

What we have to do is to evaluate **tradeoffs**

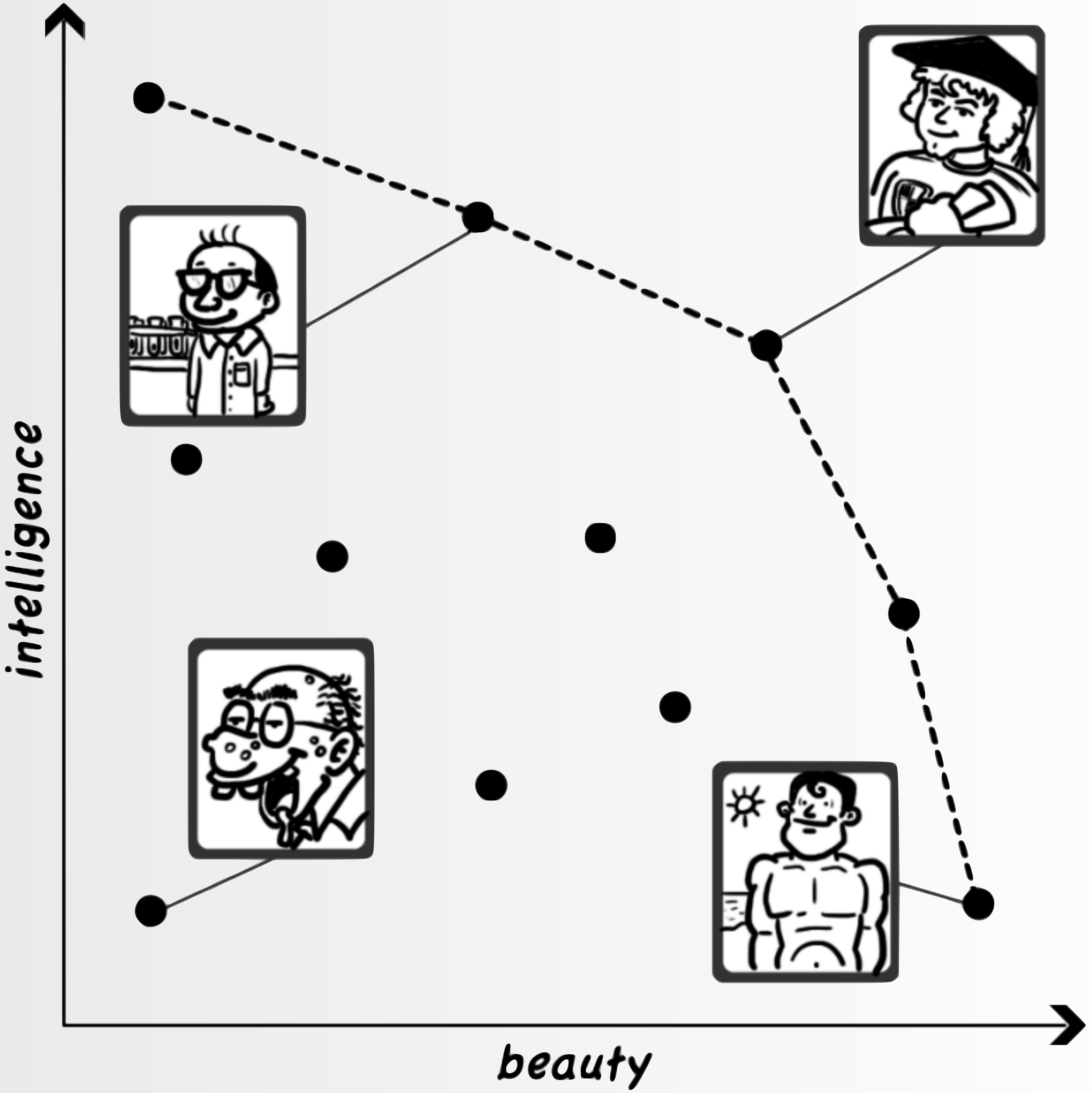
Pareto Optimality

- Define the **objective vector**:

$$\mathbf{z} = \mathbf{f}(\mathbf{x}) = \{f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})\}$$

- Consider a minimization task, an objective vector $\mathbf{z}$ is said to **dominate** $\mathbf{z}'$ if $z_k \leq z'_k \ \ \forall k$ and $\exists h$ such as $z_k < z'_k$

- A point $\hat{x}$ is **Pareto-optimal** if there is no other $x \in \Omega$ such that $\mathbf{f}(x)$ dominates $\mathbf{f}(\hat{x})$

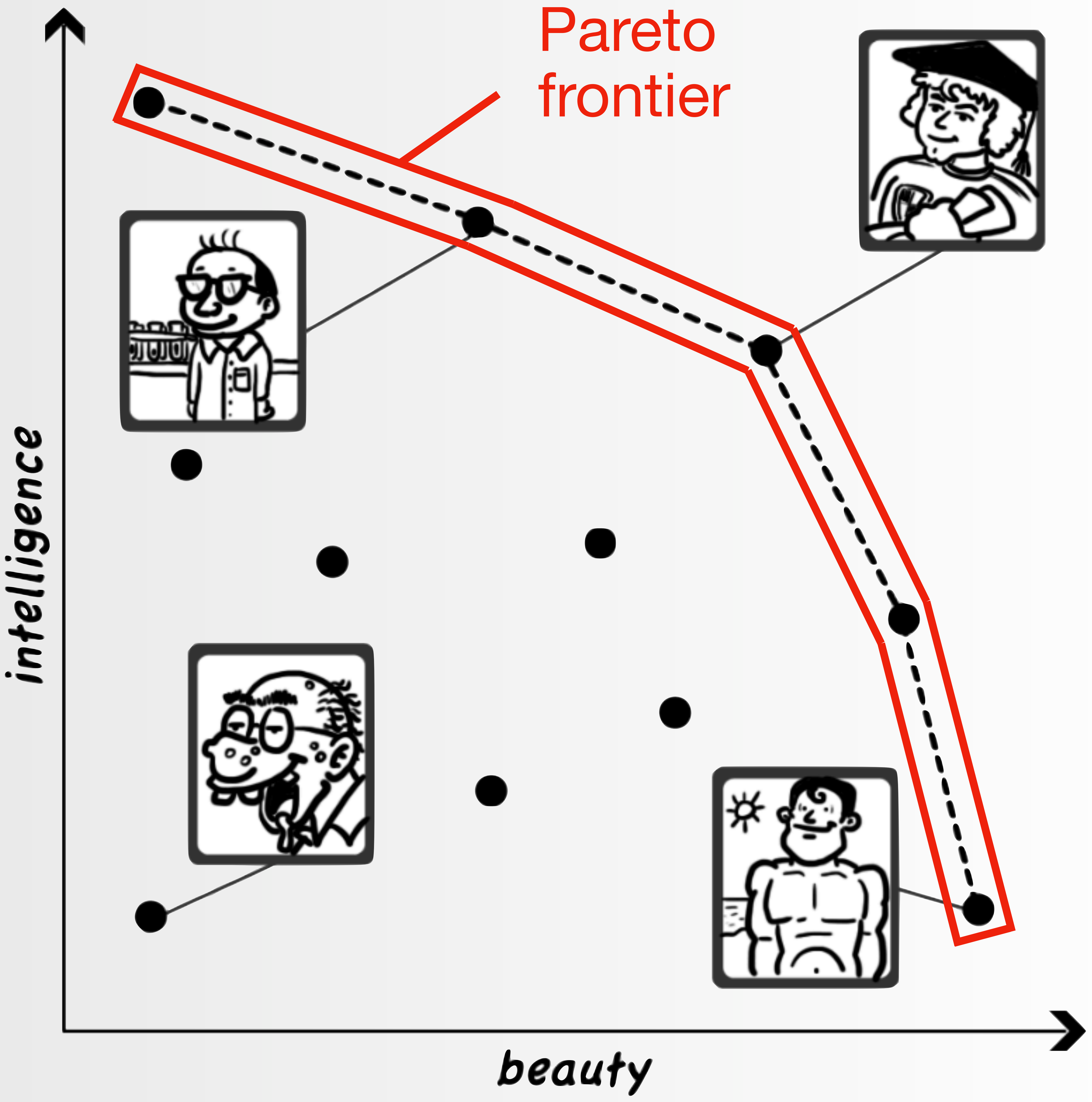


Pareto Optimality

Pareto frontier

- The **Pareto frontier** is made by the set of all the Pareto-optimal solutions

- Only on the Pareto frontier it makes sense to consider **tradeoffs**, because for points outside of it the solution would be suboptimal



Pareto Optimality

Example

Problem: find best airplane tickets (minimize price and maximize comfort)

	Price	Comfort
A	70 €	10
B	50 €	7
C	65 €	6
D	40 €	5

⇨

	Price	Comfort
A	70 €	10
B	50 €	7
C	65 €	6
D	40 €	5

B dominates C

$B_{price} \leq C_{price}$ but $B_{comfort} \geq C_{comfort}$

⇨

	Price	Comfort
A	70 €	10
B	50 €	7
D	40 €	5

Pareto frontier

Pareto Optimality

We can explicit tradeoffs between objectives and find the optimal points in the Pareto frontier applying a **combination** of the objectives.

$$g(\mathbf{x}, \mathbf{w}) = w_1 f_1(\mathbf{x}) + w_2 f_2(\mathbf{x})$$

and then

$$\begin{aligned} &\text{minimize/maximize } g(\mathbf{x}) \\ &\text{subject to } \mathbf{x} \in \Omega, \end{aligned}$$

<u>Problem:</u> weights w_1, w_2 of the linear combination are **unknown**

Multi-Objective optimization

To sum up

- MOOP consist in **multiple objectives** to optimize
- **No univocal optima** solution, need **tradeoffs**
- Pareto Optimality helps distinguish solutions which behave **better** than others
- Consider tradeoffs on the Pareto Frontier only, **undominated solutions**

Following: main Pareto optimization techniques